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THE GLOBAL STABILITY OF TWO-DIMENSIONAL SYSTEMS FOR CONTROLLING ANGULAR ORIENTATION†

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A two-dimensional dynamical system with an angular coordinate, describing the operation of a two-position automatic steering device, the plane rotations of a spacecraft controlled by an angular orientation system, and a phase-lock control frequency system with a proportional-plus-integral filter is considered. An analogue of the Barbashin–Krasovskii theorem is obtained for differential inclusions with angular coordinates. This is used to prove the global stability of two-dimensional systems of angular orientation when there is no external force moment. It is shown that, when there is a constant external force moment in phase space, a region of initial conditions exists that corresponds to circular motions of a spacecraft. © 2001 Elsevier Science Ltd. All rights reserved.

The classical example of a two-dimensional system for controlling angular orientation is the two-position automatic steering device [1]. In the traditionally assumed idealization, the equations of such a system have the form

$$I\ddot{\theta}+\alpha\dot{\theta}=M(\sigma)+M_0, \quad \sigma=\theta+b\dot{\theta} \tag{1}$$

where θ is the deviation of the vessel from the specified course, I is the moment of inertia of the vessel, α is the viscous friction coefficient, $M(\sigma)$ is the moment of the forces produced by the steering device, M_0 which here is assumed to be constant, and σ is the control signal fed to the steering-gear.

In the two-position automatic steering device, $M(\sigma)$ is a 2π- periodic function, the graph of which is shown in Fig. 1.

Here, $M(-0) = -M(+0)$ and, at the discontinuity points $\sigma = k\pi$, the values of the function $M(\sigma)$ are the intervals $[M(2k\pi + 0), M(2k\pi - 0)]$ or $[M((2k + 1)\pi - 90), M((2k + 1)\pi + 0)]$.
In this case, the solution of system (1) is defined as the solution of the differential inclusion [2]

$$I\ddot{\theta}+\alpha\dot{\theta}\in M(\sigma)+M_0, \quad \sigma=\theta+b\dot{\theta}$$

or in accordance with Filippov [3, 4]. Both these definitions for system (1) give identical solutions which coincide with the phase portraits considered earlier [1].

Equations (1) also describe the plane rotations of a spacecraft controlled by an angular orientation system [5]. Since the motion occurs in outer space, $\alpha = 0$, and as regards the control moment $M(\sigma)$ created by jet engines, it is assumed that there is an insensitivity zone $|-\Delta, \Delta|$. The graph $M(\sigma)$ in this case is presented in Fig. 2 and differs in having insensitivity zones.

Making the replacement $\eta = \dot{\theta}$, we reduce system (1) to the form

$$\dot{\eta}=-a\eta-f(\sigma), \quad \dot{\sigma}=\beta\eta-bf(\sigma) \tag{2}$$

$$a=\frac{\alpha}{I}, \ \beta=1-ab, \ f(\sigma)=\varphi(\sigma)-\gamma, \ \varphi(\sigma)=-\frac{M(\sigma)}{I}, \ \gamma=\frac{M_0}{I}$$

Thus, the parameters of system (2) satisfy the conditions $a \geqslant 0, b > 0, \beta = 1 - ab$ and $\gamma \geqslant 0$. We will also assume that $\beta \neq 0$.

Note that, when $a > 0$ and $\Delta = 0$, system (2) also describes the dynamics of phase synchronization systems with a proportional-plus-integral filter and the bang-bang characteristics of a phase-sensitive discriminator [6–8]. A detailed study of such as system showed [9] that a certain positive function

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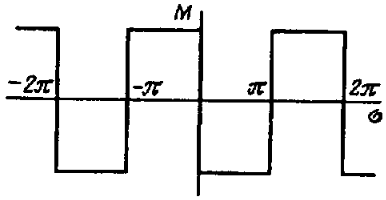

Fig. 1

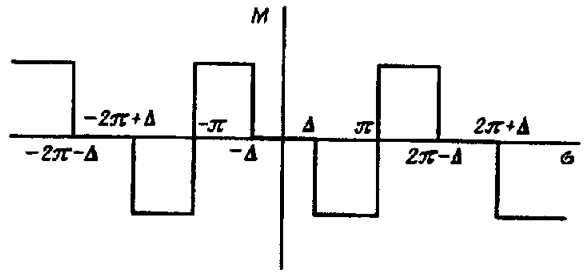

Fig. 2

$\Psi(a, b)$ exists possessing the following property: when $\gamma < \Psi(a, b)$, any solution of system (2) tends to a certain equilibrium state when $\gamma > \Psi(a, b)$ and $\gamma < 1$ both solutions approaching equilibrium states and solutions corresponding to circular motions exist (for an automatic steering device, this is infinite rotation of the vessel about the centre of gravity, and in such a case the control system is unable to damp the external force moment). Using computer calculations, the values of $\Psi(a, b)$ were calculated approximately [9].

Below it will be shown that an entirely different situation arises when $a = 0$. Here, when $\gamma = 0$. all solutions approach a stationary set. However, for any $\gamma > 0$ there are solutions corresponding to circular motions of the spacecraft.

We will examine system (2) when $\Delta \geqslant 0$.

Theorem 1. Let $\gamma = 0$. Then, any solution of system (2) tends to a certain equilibrium state.

Theorem 2. Let $\gamma > 0$ and $a = 0$. Then a positive number ε and a solution $\eta(t)$, $\sigma(t)$ of system (2) exist such that

$$\eta(t) \geqslant \varepsilon, \quad \forall t \geqslant 0 \tag{3}$$

We recall that, it follows from inequality (3) that $\dot\theta(t) \geqslant \varepsilon > 0$, $\forall t \geqslant 0$.

The proof of Theorem 1 will require the formulation of a Lyapunov-type lemma for the differential inclusions

$$\dot x \in f(x), \quad x \in R^n \tag{4}$$

where $f(x)$ is a semicontinuous vector function which maps each point $x \in R^n$ into a bounded closed convex set $f(x)$.

We recall the definitions of the terms used here [2].

Definition 1. We will say that the set $U_\varepsilon(\Omega)$ is the ε-vicinity of the set Ω if

$$U_\varepsilon(\Omega) = \left\{ x \mid \inf_{y \in \Omega} |x - y| < \varepsilon \right\}$$

where $|.|$ is the Euclidean norm in R^n.

Definition 2. The function $f(x)$ will be called semicontinuous at the point x if, for any $\varepsilon > 0$, a number $\delta(x, \varepsilon) > 0$ exists such that

$$f(y) \subset U_\varepsilon(f(x)), \quad \forall y \in U_\delta(x)$$

Definition 3. The vector function $x(t)$ is termed the solution of the differential inclusion (4), if it is absolutely continuous and, for t values for which a derivative x exists the following inclusion is satisfied

$$\dot x \in f(x(t))$$

For the differential inclusions (4) in the assumptions made here, the local theorem of the existence of a solution of Cauchy's problem holds, as well as the following theorem on the extendibility of the solutions: if, when $t \geqslant 0$ for the solutions $x(t)$, the interval $[0, T)$ is the maximum interval of definition of $x(t)$, then a sequence $t_k \in [0, T)$, $t_k \to T$ exists when $k \to \infty$ such that $x(t_k) \to \infty$ when $k \to \infty$.

These theorems were proved earlier [2].

We will now assume that, for the linearly independent vectors $d_1, \ldots, d_m$, the following equalities are satisfied

$$f(x + d_j) = f(x), \qquad \forall x \in R^n \tag{5}$$

An obvious property clearly follows from conditions (5).

Proposition 1. If $x(t)$ is the solution of inclusion (4), then, for any integer k, the sum $x(t) + kd_j$ is also the solution of inclusion (4).

We will introduce the discrete group

$$\Gamma = \left\{ x = \sum_{j=1}^{m} k_j d_j \mid k_j \in Z, \quad j = 1, \ldots, m \right\}$$

into consideration. Here, Z is a set of integers.

Consider the factor group R^n/Γ, the elements of which are classes of residues $[x] \in R^n/\Gamma$. They are defined in the following way: $[x] = \{x + u/u \in \Gamma\}$. We will introduce the so-called plane metric

$$\rho([x], [y]) = \inf_{z \in [x], v \in [y]} | z - v | \tag{6}$$

It follows from Proposition 1 that the metric space R^n/Γ thus introduced is the phase space for inclusion (4), i.e. it is divided into non-intersecting trajectories of inclusion (4).

Definition 4. The point $[p] \in R^n/\Gamma$ is termed the ω-limit point of the trajectory $[x(t)]$ if a sequence $t_k \to +\infty$ exists such that

$$[p] = \lim [x(t_k)] \text{ and } k \to \infty$$

Convergence here is understood in the sense of the metric $p([x], [y])$.

Lemma. Suppose the ω-limit set Ω of the trajectory $[x(t)]$ is bounded. Then, through each to ω-limit point of Ω at least one trajectory $[y(t)]$ passes, determined when $t \in R^1$ and consisting entirely of ω-limit points of Ω, i.e.

$$[y(t)] \in \Omega, \qquad \forall t \in R^1$$

The proof of the lemma essentially repeats the proof of Theorem 2.2.5 in [2], with the Euclidean metric replaced by metric (6).

The following theorem is an extension of the well-known Barbashin–Krasovskii theorem [10] to differential inclusions having the phase space R^n/Γ. It extends analogous results formulated earlier [11, 12].

Theorem 3. Suppose a continuous function $V(x)$: $R^n \to R^1$ exists such that the following conditions are satisfied:

1) $V(x + d) = V(x)$, $\forall x \in R^n$, $\forall d \in \Gamma$;

2) $V(x) + \sum_{j=1}^{m} (d_j^* x)^2 \to \infty$ when $|x| \to \infty$;

3) for any solution $x(t)$ of inclusion (4), the function $V(x(t))$ is non-increasing;

4) if $V(x(t) \equiv V(x(0))$, then $x(t)$ is an equilibrium state.

Then, as $t \to \infty$, any solution of inclusion (4) tends to a stationary set of this inclusion.

Note that the fact that the solution $x(t)$ tends to a stationary set Λ as $t \to \infty$ means that

$$\lim_{t \to +\infty} \inf_{z \in \Lambda} | z - x(t) | = 0$$

Proof. According to condition 1 of the theorem, it is possible to define the function $V([x])$: $R^n/\Gamma \to R^1$ in the following way: $V([x]) = V(x)$.

From condition 2 of the theorem it follows that $V([x]) \to +\infty$ as $[x] \to \infty$.

Using this property, we will prove that any trajectory $[x(t)$ in the space R^n/Γ is bounded when $t \geq 0$. Assuming the opposite, we obtain that an increasing sequence t_k exists such that $[x(t_k)] \to \infty$ as $k \to \infty$. It follows that $V([x(t_k)]) \to +\infty$ as $k \to \infty$. However, this contradicts condition 3 of the theorem.

Thus, the trajectory $[x(t)]$ is bounded when $t \geq 0$ and, consequently, defined in $[0, +\infty)$. From the boundedness of the trajectory $[x(t)]$ it follows that its ω-limiting set Ω is also bounded. Let $[p] \in \Omega$. Then, using the lemma, we obtain that a trajectory $[y(t)]$ exists such that

$$[y(0)] = [p], \quad [y(t)] \in \Omega, \quad \forall t \in R^1$$

Since, according to condition 3 of the theorem. $V([x(t)])$ is a non-increasing function, and the boundedness in $[0, +\infty)$ of the function $V([x(t)])$ follows from the boundedness of $[x(t)]$ and the continuity of $V([x])$, we have that the following limit exists

$$\lim_{t \to +\infty} V([x(t)]) = V_0$$

However, then $V([y(t)]) = V_0$, $\forall t \in R^1$. From this and from condition 4 of the theorem, it follows that $y(t) = y(0)$, $\forall t \in R^1$.

Thus, the ω-limit set Ω consists of the states of equilibrium of inclusion (4).
The statement of the theorem also follows from this.

Proof of Theorem 1. Consider the function

$$V(\eta, \sigma) = \eta^2 + 2q\int_0^\sigma g(\sigma)d\sigma, \quad q = \frac{1+ab}{(1-ab)^2}$$

where $g(\sigma)$ is a certain single-valued function, identical with $f(\sigma)$ at the points where the function $f(\sigma)$ is single valued. For t values for which $\sigma(t)$ is a point where $f(\sigma)$ is single valued, the following relation holds

$$\frac{d}{dt}V(\eta(t), \sigma(t)) = -2a\left(\eta(t) - \frac{b}{1-ab}f(\sigma(t))\right)^2 - \frac{2b}{(1-ab)^2}f(\sigma(t))^2 \tag{7}$$

In sliding modes of system (2) we have $\sigma(t) = \sigma^*$, where σ^* is the point where $f(\sigma)$ is single valued. Therefore, from the second equation of system (2) we have

$$\beta\eta(t) - b\xi(t) = 0 \tag{8}$$

where $\xi(t)$ is a so-called extended function [2]. In this case, the first equation of system (2) will take the form $\dot{\eta} = -a\eta - \xi(t)$. Substituting into this equation the function $\xi(t)$ found from (8), and taking into account the identity $\sigma(t) = \sigma^*$, we obtain that, in the sliding mode.

$$\frac{d}{dt}V(\eta(t), \sigma(t)) = -2\left(a + \frac{\beta}{b}\right)\eta(t)^2 \tag{9}$$

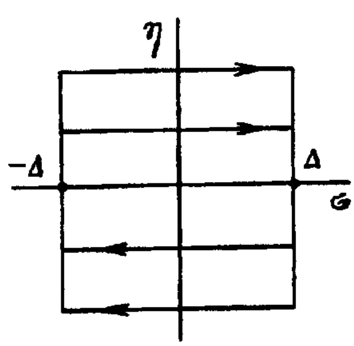

Fig. 3

From relations (7) and (9) it follows that, for the function $V(\eta, \sigma)$, condition 3 of Theorem 3 is satisfied. It is also clear that, when $a > 0$, the satisfaction of condition 4 of Theorem 3 follows from relations (7) and (9). When $a = 0$, in the bands $\{\eta \in R^1, \sigma \in (2k\pi - \Delta, 2k\pi + \Delta)\} = 0$ the relation $V(\eta(t), \sigma(t)) = \text{const}$ is satisfied. However, it is easy to show by direct integration of system (2) that whole trajectories positioned in these bands are only states of equilibrium. For any other trajectory, only some part of it can lie in the bands $\{\eta \in R^1, \sigma \in (2k\pi - \Delta, 2k\pi + \Delta)\}$ (Fig. 3). From this, and from (7) and (9), it follows that condition 4 of Theorem 3 is satisfied when $a = 0$. Conditions 1 and 2 of the theorem are obviously satisfied by virtue of the condition $\gamma = 0$. It is clear that in this case $m = 1$ and

$$\int_0^{2\pi} g(\sigma)d\sigma = 0, \quad d_1 = \begin{pmatrix} 0 \\ 2\pi \end{pmatrix}$$

Thus, all the conditions of Theorem 3 are satisfied, and any solution of system (2) tends, as $t \to +\infty$, to a stationary set.

Elementary analysis of the behaviour of trajectories in the vicinity of the stationary set of system (2) enables to make the somewhat stronger assertion that, when $t \to +\infty$, any solution tends to a certain equilibrium state.

Proof of Theorem 2. We will use the Chaplygin–Kamke comparison principle [12–14], by constructing the comparison system

$$\dot{u} = -f_1(\vartheta), \quad \dot{\vartheta} = u - bf_1(\vartheta)$$
$$f_1(\theta) = \begin{cases} \varphi(\theta) - \gamma_1 & \text{when } \theta \in (-\pi, -\Delta) \text{ and } \theta \in (\Delta, \Delta) \\ 0 & \text{when } \theta \in (-\Delta, \pi) \end{cases} \tag{10}$$

The positive number $\gamma_1 \in (0, \gamma)$ will be determined later. We will find the periodic solution $u_0(\vartheta)$ of the first-order equation

$$\frac{du}{d\vartheta} = \frac{-f_1(\vartheta)}{u - bf_1(\vartheta)} \tag{11}$$

equivalent to system (10), for which

$$u_0(\vartheta) > b \max_{\vartheta} f_1(\vartheta) \tag{12}$$

From the comparison principle we will then establish that any solution of system (2) with initial data $\eta(0)$, $\sigma(0)$ satisfying the condition $\eta(0) > u_0(\sigma(0))$ will possess the following property

$$\eta(t) > u_0(\sigma(t)) \tag{13}$$

Relations (12) and (13) prove Theorem 2.

We will now clarify the conditions for the solution $u_0(\vartheta)$ to exist. From the integration of Eq. (11) it is clear that

$$-f_1^-(\pi - \Delta) = \frac{1}{2}(u_0(-\Delta)^2 - u_0(-\pi)^2) - bf_1^-(u_0(-\Delta) - u_0(-\pi)) \tag{14}$$

$$-f_1^+(\pi - \Delta) = \frac{1}{2}(u_0(\pi)^2 - u_0(\Delta)^2) - bf_1^+(u_0(\pi) - u_0(\Delta)) \tag{15}$$

where f_1^- is the value of $f_1(\vartheta)$ in the interval $(-\pi, -\Delta)$, and f_1^+ is the value of $f_1(\vartheta)$ in the interval (Δ, π). Summing (14) and (15) and taking into account that $u_0(\pi) = u_0(-\pi)$ and $u_0(-\Delta) = u_0(\Delta)$, we obtain

$$\gamma_1(\pi - \Delta) = b\varphi^-(u_0(\pi) - u_0(\Delta)) \tag{16}$$

where φ^- is the value of $\varphi(\vartheta)$ in $(-\pi, -\Delta)$.

From (15) and (16) we obtain the following equality for determining $u_0(\pi)$

$$u_0(\pi) = -\frac{b\varphi^- f_1^+}{\gamma_1} + \frac{\gamma_1(\pi - \Delta)}{2b\varphi^-} + bf_1^+$$

It is clear that, for sufficiently small γ_1, the inequality $u_0(\vartheta) \geqslant u_0(\pi) > bf_1^+$ is satisfied. Inequality (12) follows from this. The theorem is proved.

Various extensions of Theorems 1 and 2 both to wider classes of non-linearities and to multi-dimensional dynamical systems are possible within the framework of the frequency methods described in [8, 12, 15].

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